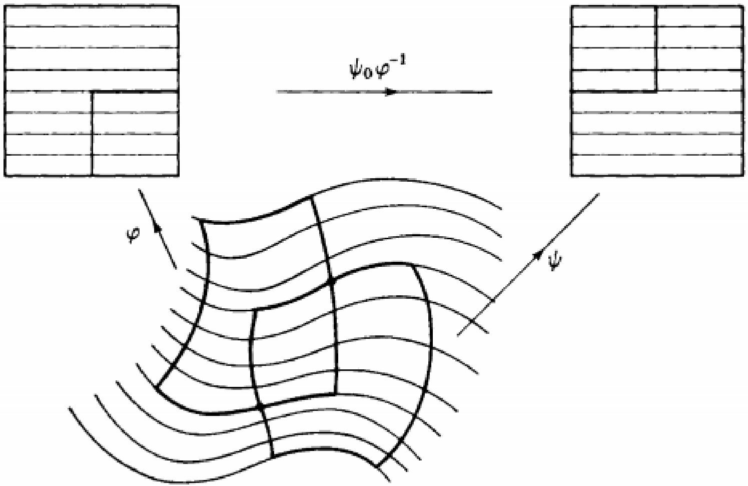
Background story: holomorphic foliation

$X :=$ smooth complex manifold.
A **holomorphic foliation** $\mathcal{F}$ on X is a collection of charts:
$\{\phi_i : U_i^{\subset X} \to V_i^{\subset \mathbb{C}^k} \times W_i^{\subset \mathbb{C}^{n-k}}\}$, s.t.
$\phi_i \circ \phi_j^{-1}(x, y) = (f(x), g(x, y))$.

A **leaf** of $\mathcal{F}$ is an immersed submanifold $L \subset X$ which has locally constant coordination w.r.t. V_i.

Ex: $f : X \to Y$ a submersion of complex mainfolds.
Leaves: $f^{-1}(p)$, $p \in Y$



$\psi_0\varphi^{-1}$
φ
ψ

Frobenius theorem

A foliation $\mathcal{F}$ on X gives rise to a sub-bundle $T_{\mathcal{F}} \subset T_X$ of holomorphic tangent bundle, which is **closed under** $[-,-]$. Conversely:

Theorem

A sub-bundle $T' \subset T_X$ closed under $[-,-]$ gives rise to a holomorphic foliation on X.

Alternative description:

A quotient bundle $\Omega_X \twoheadrightarrow \Omega_{\mathcal{F}}$ s.t. there is a map of cdga's

$$\mathbf{DR}(X) \twoheadrightarrow \left(\oplus^i \wedge^i \Omega_{\mathcal{F}}[i], \bar{d}_{DR} \right) := \mathbf{DR}(\mathcal{F})$$

Algebraic analogue

Lie algebroid

$\mathfrak{g} \hookrightarrow T_{X/\mathbb{C}}$

~~locally free~~ $\mathcal{O}_X$ module

closed under $[-,-]$

$\overset{?}{\leftrightsquigarrow}$

Algebraic foliation

$\Omega_{X/\mathbb{C}} \twoheadrightarrow \Omega_{\mathscr{F}}$ quotient ~~bundle~~,

inducing maps of cdga's

$\mathrm{DR}(X/\mathbb{C}) \twoheadrightarrow (\oplus_{i\geq 0} \wedge^i \Omega_{\mathscr{F}}[i], \bar{d}_{DR})$

Ex: Maximal destabilzing subsheaves of tangent bundles.

Question 1: How to compare (singular) Lie algebroids and algebraic foliations?

Integrability

Fact: not every algebraic foliation has algebraic leaf.
Consider $\frac{dx}{x} + dy$ on $\mathbb{A}^2 - \{(0, y)\}$.

Question 2: How to say something about leaves?
Question 3: How about positive characteristics?

Adv. of Infinitesimal derived foliation

Features:

- a derived analogue of algebraic foliation **over any SCR**
- enjoy **formal leaves** described by formal moduli problems
- under some finiteness condition, equivalent to a sub-∞-category of **partition Lie algebroids**
- many examples

Abbr: Inf.foliation for short.

Tool: Refined Koszul duality.

Ex: Purely inseparable Galois Theory

Theorem (Brantner-Waldron)

Consider K/F a finite purely inseparable field extension in char$p > 0$. Then there is a cat.equivalence bewteen

$$\{\textit{intermediate fields } K \supset E \supset F, \textit{ with morphism } \supset\}$$
$$\Updownarrow$$
$$\textit{full subcategory } \mathcal{C} \subset \mathrm{LieAlgd}^{\pi}_{R/k,\Delta}$$

where $\mathcal{C}$ is spanned by $\mathfrak{g} \to \mathbb{L}^{\vee}_{K/F}[1]$ s.t.

- *the anchor map $\pi_1(g) \to Der_F(K)$ is* **injective**
- $\pi_i = 0$ *for* $i \neq 0, 1$
- $dim_K(\pi_0(\mathfrak{g})) = dim_K(\pi_1(\mathfrak{g})) < +\infty$

Derived algebra monad: LSym

$\mathrm{Sym}^{\heartsuit} \curvearrowright$ **Ab** has non-commutative derived functor LSym^{cn} acting on $\mathrm{Mod}_{\mathbb{Z}}^{cn}$, which is the monad defining SCR's.

Theorem (Raksit, Brantner-Campos-Nuiten)

There is a unique sifted-colimit-preserving endo-functor $\mathrm{LSym} \in \mathit{End}(\mathrm{Mod}_{\mathbb{Z}})$ *extending* LSym^{cn}*. Additionally, it is endowed with a canonical monad structure.*

Remark

LSym also naturally acts on

$$\mathrm{Fil}\,\mathrm{Mod}_{\mathbb{Z}} := \mathrm{Fun}(\mathbb{Z}_{\geq}, \mathrm{Mod}_{\mathbb{Z}})$$
$$\mathrm{Gr}\,\mathrm{Mod}_{\mathbb{Z}} := \mathrm{Fun}(\mathbb{Z}_{dist}, \mathrm{Mod}_{\mathbb{Z}})$$

Infinitesimal cohomology

Write $\mathsf{Alg}_{\mathrm{LSym}}(-)$ as $\mathsf{DAlg}(-)$. Observe that

$$\mathsf{Mod}_{\mathbb{Z}} \xleftarrow{F^0} \mathsf{Fil}_{\geq 0}\,\mathsf{Mod}_{\mathbb{Z}} \xrightarrow{\mathrm{Gr}} \mathrm{Gr}_{\geq 0}\,\mathsf{Mod}_{\mathbb{Z}}$$

are LSym-linear.

Fact: for $k \in \mathsf{SCR}$,

$$\mathrm{Gr}^0 : \mathsf{DAlg}(\mathsf{Fil}_{\geq 0}\,\mathsf{Mod}_k) \to \mathsf{DAlg}(\mathsf{Mod}_k)$$

admits a fully faithful left adjoint $\mathsf{F}^{*}_{\mathsf{H}}\mathbb{\Pi}_{-/k}$, called **Hodge filtered infinitesimal cohomology.**

$$\mathrm{Gr}(\mathsf{F}^{*}_{\mathsf{H}}\mathbb{\Pi}_{R/k}) \simeq \mathrm{LSym}_R(\mathbb{L}_{R/k}[-1])$$

Function rings of inf.foliations

Set $\mathcal{D}^{\mathrm{Fil}}_{R/k}$ by a homotopy pullback of ∞-category

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{Fil}}_{R/k} & \longrightarrow & \mathrm{DAlg}(\mathrm{Fil}_{\geq 0}\,\mathrm{Mod}_k) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Gr}^0} \\ \{R\} & \longrightarrow & \mathrm{DAlg}(\mathrm{Mod}_k) \end{array}$$

Definition

$A \in \mathcal{D}^{\mathrm{Fil}}_{R/k}$ is said to be foliation like if it is complete and

$$\mathrm{Gr}(A) \simeq \mathrm{LSym}_R(\mathrm{Gr}^1(A)).$$

Here $\mathrm{Gr}^1(A)$ is called the cotangent complex of A.

Ex: $\widehat{\mathrm{F}^*_{\mathrm{H}}\sqcap_{\mathrm{R/k}}}$ is the initial foliation-like algebra for R/k

Geometry of filtration and grading

Theorem (Moulinos)

$$\mathrm{QC}(B\mathbb{G}_m) \simeq \mathrm{Gr}\, Sp, \qquad \mathrm{QC}([\mathbb{A}^1/\mathbb{G}_m]) \simeq \mathrm{Fil}\, Sp$$

Definition

$\mathbb{G}_m - \mathbf{dSt}_k$ is called ∞-category of graded derived stacks.

Ex: For each $E \in \mathrm{Mod}_k$, the *linear stack* $\mathbb{V}(E)$ of E

$$\mathbb{V}(E)(A) := \mathrm{Map}_{\mathrm{Mod}_k}(E, A) \simeq \mathrm{Map}_{\mathrm{DAlg}_k}(\mathrm{LSym}_k\, E, A)$$

is a graded stack by multiplication of k on E.

Graded mixed structure

$\mathbb{D}_{-}^{\vee} := \mathrm{LSym}_k(k[1]) \simeq k \oplus k[1]$ can be regarded as a **graded Hopf derived algebra**, putting $k[1]$ at weight -1.
$\Omega_o\mathbb{G}_a = \mathsf{Spec}(\mathbb{D}_{-}^{\vee})$ admits a $\mathbb{G}_m$-action by canonical multiplication.

$$\mathcal{H}_\pi := \mathbb{G}_m \ltimes \Omega_o\mathbb{G}_a$$

Theorem

$$\mathsf{QC}(B\,\mathcal{H}_\pi) \simeq \mathit{LComod}_{\mathbb{D}_{-}^{\vee}}(\mathrm{Gr}\,\mathsf{Mod}_k) =: \mathsf{DG}_{-}\,\mathsf{Mod}_k$$

Fact: $\mathsf{DG}_{-}\,\mathsf{Mod}_k \simeq \mathsf{Fil}^{\mathsf{cpl}}\,\mathsf{Mod}_k$, furthermore

$$\mathsf{DG}_{-}\,\mathsf{DAlg}_k := \mathsf{LComod}_{\mathbb{D}_{-}^{\vee}}(\mathrm{Gr}\,\mathsf{DAlg}_k) \simeq \mathsf{Fil}^{cpl}\,\mathsf{DAlg}_k$$

Infinitesimal derived foliation

Definition (Toën-Vezzosi)

$X := \mathrm{Spec}(R)$ over k. An inf.foliation consists of:

- a $\mathcal{H}_\pi$-stack $\mathcal{F}$ over x
- a $\mathbb{G}_m$ equivaraint map $\mathcal{F} \to X$, which is equivalent to $\mathbb{V}(E) \to X$ the projection of some R-**linear stack**.

Realizing inf.foliation

Theorem

There is a fully faithful embedding

$$Spec^{nc} : \mathcal{D}^{\mathcal{F}\mathrm{ol}}_{\mathrm{R/k,b}} \hookrightarrow (\mathcal{F}ol^{\pi}_{R/k})^{op}.$$

Ex: $\mathrm{Spec}^{nc}(\widehat{\mathrm{F}^{*}_{\mathrm{H}}\sqcap_{\mathrm{R/k}}})$ is the final inf.foliation on $\mathrm{Spec}(R)$,

$$\mathcal{L}^{\mathrm{gr}}_{\pi}(X/k) := \mathrm{Map}_{\mathbf{dSt}_k}(\Omega_o\mathbb{G}_a, X)$$

Naïve attemption to Koszul duality

$$(\mathrm{DAlg}_k)_{/R} \underset{\mathrm{sqz}}{\overset{\mathrm{cot}_{k//R}}{\rightleftarrows}} \mathrm{Mod}_R \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} (\mathrm{Mod}_R)^{op}$$

gives a monad $T_{\text{naïve}} := (\mathrm{cot}_{k//R} \circ \mathrm{sqz}(-)^\vee)^\vee$, but not s.c.p.

Question: Modify $T_{\text{naïve}}$ to have a s.c.p. monad, generalizing Kosuzl duality between commutative algebra and Lie algebra.

Definition

An ∞-category $\mathscr{A}$ is said to be **additive** if it has finite direct sums and $Ho(\mathscr{A})$ is additive. It has the followng module categories:

- $\mathrm{Mod}_{\mathscr{A}} := Sp(\mathscr{P}_{\Sigma}(\mathscr{A}))$
- $\mathrm{APerf}_{\mathscr{A}} \subset \mathrm{Mod}_{\mathscr{A}}$ generated by $\mathscr{A}$ under geom.realizations
- $\mathrm{Coh}_{\mathscr{A}} \subset \mathrm{APerf}_{\mathscr{A}}$ spanned by eventually coconnective obj.

Definition

$\mathscr{A}$ is said to be coherent if $\mathrm{APerf}_{\mathscr{A}}$ (and $\mathrm{APerf}_{\mathscr{A}^{op}}$)inherits the t-structure from $\mathrm{Mod}_{\mathscr{A}}$ ($\mathrm{Mod}_{\mathscr{A}^{op}}$). In this case, the ∞-category of **pro-coherent** $\mathscr{A}$-modules is defined to be

$$\mathrm{QC}^{\vee}_{\mathscr{A}} := \mathrm{Ind}(\mathrm{Coh}^{op}_{\mathscr{A}^{op}})$$

Ex: modules over coherent SCR R

There is an adjunction

$$\iota : \mathsf{Mod}_R \rightleftarrows \mathsf{QC}_R^\vee$$

exhibits Mod_R as the right completion of $\mathsf{QC}_R^\vee$.
If R is eventually coconnective, $\mathrm{Perf}_R \subset \mathsf{Coh}_R$, thus $\mathsf{Mod}_R \subset \mathsf{QC}^\vee$.

The same holds for filtered/graded modules.

Ex: derived ∞-operad

Let R be ordinary comm.ring
Symmetric sequences: $\mathrm{sSeq}_R := \mathsf{Fun}(B\Sigma, \mathsf{Mod}_R)$.
Ex: hom_E such that $hom_E(n) = \mathrm{hom}(E^{\otimes n}, E)$

Set $R[\mathcal{O}_\Sigma] \subset \mathrm{sSeq}_R^{\heartsuit}$ spanned by $R[G/H]$'s and direct sums.
Lemma. $R[\mathcal{O}_\Sigma]$ is a coherent additive ∞-category.

Definition

$\mathrm{sSeq}_R^{\vee} := \mathsf{QC}^{\vee}_{R[\mathcal{O}_\Sigma]}$ is called the ∞-categoryof **pro-coherent genuine symmetric sequences**

Extension of operations

Theorem (*[BCN, Theorem 2.49]*)

There is a natural transformation of symmetric monoidal functors

$$\iota : \mathrm{Mod} \Rightarrow \mathrm{QC}^{\vee} : \mathcal{A}dd^{coh,poly} \to \mathcal{P}r^{st,\Sigma}$$

sending each polynomial functor to its **right-left derived functor***.*

Proposition. There is a s.c.p. functor $SCR \to \mathcal{A}dd$ extending $R^{dist} \mapsto R[\mathcal{O}_{\Sigma}]$.

$\mathrm{sSeq}_R^\vee$ has various monoidal structures

- Two symm.monoidal structure:
  $\otimes$ by Day convolution, $\otimes_{\mathrm{lev}}$
- Two composite tensor:

$$X \circ Y \simeq \bigoplus_{r \geq 0} (X(r) \otimes_{\underline{R}} Y^{\otimes r})_{\Sigma_r}$$

$$X \,\bar{\circ}\, Y \simeq \bigoplus_{r \geq 0} (X(r) \otimes_{\underline{R}} Y^{\otimes r})^{\Sigma_r}$$

- Indentifying $\mathrm{QC}_R^\vee$ with arity 0 symm.seq., both $(\mathrm{sSeq}_R^\vee, \circ)$ and $(\mathrm{sSeq}_R^\vee, \bar{\circ})$ acts on $\mathrm{QC}_R^\vee$

Ex: $\mathrm{Com}(n) = R[G/G] = R$ gives an algebra obj for both $\circ$ and $\bar{\circ}$, defining derived comm.algebra and **derived divided power algebra** resp.

Linear dual

Proposition.
The linear dual $(-)^\vee$ is extended to $\mathrm{QC}_R^\vee$, and gives rise to

$$\mathrm{APerf}_R \simeq (\mathrm{APerf}_R^\vee)^{op}$$

Proposition.
Taking linear dual w.r.t $\otimes_{\mathrm{lev}}$ is lax-monoidal from $\circ$ to $\bar{\circ}$. Specially, it takes ∞-cooperads for $\circ$ to ∞-operads for $\bar{\circ}$.

Left-right derived functor:

Let $\mathscr{A}$ be a coherent additive ∞-category and $\mathcal{V}$ an ∞-category with sifted colimits. Then the restriction determines a commutative square

$$\begin{array}{ccc} \mathrm{Fun}_{\Sigma}(\mathrm{QC}^{\vee}_{\mathscr{A}}, \mathcal{V}) & \xrightarrow{\simeq} & \mathrm{Fun}_{\sigma, reg}(\mathrm{APerf}^{\vee}_{\mathscr{A}, \leqslant 0}, \mathcal{V}) \\ \downarrow{\scriptstyle \iota^*} & & \downarrow \\ \mathrm{Fun}_{\Sigma}(\mathrm{Mod}_{\mathscr{A}}, \mathcal{V}) & \xrightarrow[\simeq]{} & \mathrm{Fun}_{\sigma}(\mathrm{Perf}^{\vee}_{\mathscr{A}, \leqslant 0}, \mathcal{V}) \end{array}$$

where the horizontal functors have inverses given by left Kan extensions.

Categorical bar-cobar adjunction

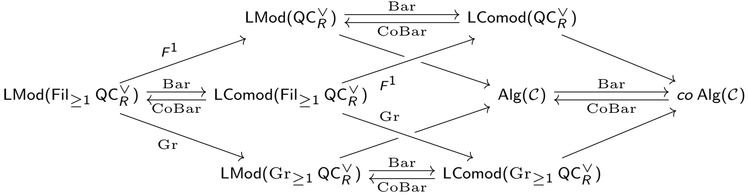
Theorem (Lurie, BCN)

Let $\mathcal{C}$ be a pointed symm.monoidal ∞-category, $\mathcal{M}$ is lefted tensored over $\mathcal{C}$. Suppose both admits geom.realization and totalization. Then,

$$\begin{array}{ccc} LMod(\mathcal{M}) & \underset{co\,\mathrm{Bar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & LComod(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathsf{Alg}(\mathcal{C}) & \underset{co\,\mathrm{Bar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathsf{Alg}(\mathcal{C}) \end{array}$$

here Bar *preserves coCartesian edges.*

$$\mathcal{C} := \left(\mathrm{sSeq}_R^\vee, \circ\right)$$



Refined Koszul duality

Fact: $\mathcal{P} = \mathsf{Com}^{nu}$ satisfies (†) **reduced** and $\mathrm{Bar}(\mathsf{Com}^{nu})$ is **almost perfect** in $\mathrm{sSeq}_R^{\vee}$.

$$
\begin{array}{ccccc}
\mathsf{Alg}_{\mathcal{P}}(\mathsf{QC}_R^{\vee}) & \underset{\mathrm{CoBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathsf{Alg}_{\mathrm{Bar}(\mathcal{P})}(\mathsf{QC}_R^{\vee}) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^{\vee}}{\rightleftarrows}} & \mathsf{Alg}_{\mathrm{KD}^{\mathsf{pd}}(\mathcal{P})}(\mathsf{QC}_R^{\vee})^{op} \\
\mathrm{adic}\downarrow\uparrow F^1 & & (-)^1\downarrow\uparrow F^1 & & \mathrm{const}\downarrow\uparrow\mathrm{colim} \\
\mathsf{Alg}_{\mathcal{P}}(\mathsf{Fil}_{\geq 1}\,\mathsf{QC}_R^{\vee}) & \underset{\mathrm{CoBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathsf{Alg}_{\mathrm{Bar}(\mathcal{P})}(\mathsf{Fil}_{\geq 1}\,\mathsf{QC}_R^{\vee}) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^{\vee}}{\rightleftarrows}} & \mathsf{Alg}_{\mathrm{KD}^{\mathsf{pd}}(\mathcal{P})}(\mathsf{Fil}_{\leq -1}\,\mathsf{QC}_R^{\vee})^{op} \\
\mathrm{Gr}\downarrow\uparrow & & \mathrm{Gr}\downarrow\uparrow & & \mathrm{Gr}\downarrow\uparrow \\
\mathsf{Alg}_{\mathcal{P}}(\mathrm{Gr}_{\geq 1}\,\mathsf{QC}_R^{\vee}) & \underset{\mathrm{CoBar}}{\overset{\mathrm{Bar}}{\rightleftarrows}} & co\,\mathsf{Alg}_{\mathrm{Bar}(\mathcal{P})}(\mathrm{Gr}_{\geq 1}\,\mathsf{QC}_R^{\vee}) & \underset{(-)^{\mathsf{i}}}{\overset{(-)^{\vee}}{\rightleftarrows}} & \mathsf{Alg}_{\mathrm{KD}^{\mathsf{pd}}(\mathcal{P})}(\mathrm{Gr}_{\leq -1}\,\mathsf{QC}_R^{\vee})^{op}
\end{array}
$$

Definition

(1) $\mathsf{Lie}^{\pi}_{R,\Delta} := \mathrm{KD}^{\mathsf{pd}}(\mathsf{Com}^{nu})$ is called the PD ∞-operad of **derived partition Lie algebra**.

(2) Filtered Chevalley-Eilenberg complex:

$\widetilde{\mathsf{CE}} : \mathsf{Alg}_{\mathsf{Lie}^{\pi}_{R,\Delta}}(\mathsf{QC}_R^{\vee}) \hookrightarrow \mathsf{Alg}_{\mathsf{Lie}^{\pi}_{R,\Delta}}(\mathsf{Fil}_{\leq -1}\,\mathsf{QC}_R^{\vee}) \to (\mathsf{DAlg}^{aug}(\mathsf{Fil}_{\geq 1}\,\mathsf{QC}_R^{\vee}))^{op}$

Koszul duality of partition Lie algebras

Theorem

Let R be an eventually coconnective coherent SCR, then $\widetilde{\mathsf{CE}}$ gives a fully faithful embedding

$$\widetilde{\mathsf{CE}} : \mathsf{Lie}^{\pi,\mathsf{dap}}_{R,\Delta} \hookrightarrow \mathcal{D}^{\mathcal{F}ol,op}_{R/R,b},$$

the essential image is spanned by A such that $\mathrm{Gr}^1(A)$ is almost perfect over R.

Return to $T_{\text{naïve}}$

Fix $k \to R$ coherent SCR's s.t. $\mathbb{L}_{R/k}$ is almost perfect

$$(\mathrm{DAlg}_k^{pro-coh})_{/R} \underset{\mathrm{sqz}}{\overset{\mathrm{cot}_{k//R}}{\rightleftarrows}} \mathrm{QC}^{v}\, ee_R \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} (\mathrm{QC}_R^\vee)^{op}$$

Observation: $\bar{T}$ fits into the fibre sequence (tested on $\mathrm{APerf}_R^\vee$)

$$\mathrm{Lie}^{\pi}_{R,\Delta}(-) \to \bar{T}(-) \to \bar{T}(0) \simeq \mathbb{L}^\vee_{R/k}[1]$$

since $\mathrm{Lie}^{\pi}_{R,\Delta}$ is **dually connective** and **dually almost perfect**, thus by right-left extension, it gives rise to a **s.c.p monad** T acting on $\mathrm{QC}_R^\vee$.

Partition Lie algebroid

Definition

$\mathrm{LieAlgd}^{\pi}_{R/k,\Delta} := \mathrm{Alg}_T(\mathrm{QC}^{\vee}_R)$ is called the ∞-category of partition Lie algebroids.

Rk. $\mathfrak{g} \in \mathrm{LieAlgd}^{\pi}_{R/k,\Delta}$ is roughly

$$\mathbb{L}^{\vee}_{R/k} \to \mathfrak{h} \to \mathfrak{g} \to \mathbb{L}^{\vee}_{R/k}[1]$$

Proposition.

$$\begin{array}{ccc}
\mathrm{LieAlgd}^{\pi,\mathrm{Fil}}_{R/k,\Delta} & \underset{\mathfrak{D}_{\mathrm{Fil}}}{\overset{C^*_{\mathrm{Fil}}}{\rightleftarrows}} & (\mathcal{D}^{\mathrm{Fil}}_{R/k})^{op} \\
\mathrm{colim} \downarrow \uparrow \mathrm{const} & & F^0 \downarrow \uparrow \mathrm{adic}' \\
\mathrm{LieAlgd}^{\pi}_{R/k,\Delta} & \underset{\mathfrak{D}}{\overset{C^*}{\rightleftarrows}} & (\mathrm{DAlg}(\mathrm{QC}^{\vee}_k)/R)^{op}
\end{array}$$

where $\mathit{fib} \circ \mathfrak{D}_{\mathrm{Fil}}(A) \simeq (\mathrm{cot}_{k//R}(R)^{\vee} \to \mathrm{cot}_{k//R}(A)^{\vee})$.

Main theorem

Theorem

Let $k \to R$ be coherent SCR's s.t. $\mathbb{L}_{R/k}$ is almost perfect, then $\widetilde{C}^* := C^*_{\mathrm{Fil}} \circ \mathrm{const}$ induces a fully faithful embedding

$$\widetilde{C}^* : \mathrm{LieAlgd}^{\pi,\mathrm{dap}}_{R/k,\Delta} \hookrightarrow (\mathcal{D}^{\mathcal{F}\mathrm{ol}}_{\mathrm{R/k,b}})^{op},$$

essential image spanned by A s.t. $\mathrm{Gr}^1(A)$ is almost perfect.

Theorem (Main)

Furtherly assume k is eventually coconnective,

$$Spec^{nc} \circ \widetilde{C}^* : \mathrm{LieAlgd}^{\pi,\mathrm{dap}}_{R/k,\Delta} \hookrightarrow \mathcal{F}ol^{\pi}_{R/k}.$$

Formal deformation

Definition

Let $k \to R$ be as before:

(1) $\mathcal{D}_R^{sm}$: **small derived algebras** is the minimal $\subset SCR_{k//R}$ s.t.:

- $\mathcal{D}_R^{sm}$ contains $\mathrm{sqz}(M)$ for all $M \in \mathsf{Coh}_{R,\geq 1}$;
- for each each diagram $A \to \mathrm{sqz}(M) \leftarrow R$ in $\mathcal{D}_R^{sm}$ with $M \in \mathsf{Coh}_{R,\geq 1}$, the pullback $A \times_{\mathrm{sqz}(M)} R$ lies in $\mathcal{D}_R^{sm}$ as well.

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(2) A **formal moduli problem** F over R (relative to k) is a functor $F : \mathcal{D}_R^{sm} \to \mathcal{S}$ such that

- F sends the final object R to a contractible space $F(R) \simeq *$;
- for each diagram $A \to \mathrm{sqz}(M) \leftarrow R$ in $\mathcal{D}_R^{sm}$ with $M \in \mathsf{Coh}_{R,\geq 1}$, the pullback $A \times_{\mathrm{sqz}(M)} R$ is taken to a pullback $F(A) \times_{F(\mathrm{sqz}(M))} *$ in $\mathcal{S}$.

Formal leaf of inf.foliation

Proposition. For each $\mathfrak{g} \in \mathrm{LieAlgd}^{\pi}_{R/k,\Delta}$,

$$MC(\mathfrak{g})(A) = \mathrm{Map}_{\mathrm{LieAlgd}^{\pi}_{R/k,\Delta}}(\mathfrak{D}(A), \mathfrak{g})$$

defines a formal moduli problem.

For bounded below inf.foliation $\mathcal{F}$, its **formal leaf** is defined as

$$FL(\mathcal{F}) := MC(\mathrm{colim}\, \mathfrak{D}_{\mathrm{Fil}}(A))$$

where A is its foliation-like algebra.
Proposition. If $\mathfrak{g} \in \mathrm{LieAlgd}^{\pi,\mathrm{dap}}_{R/k,\Delta}$, then $MC(\mathfrak{g}) \simeq FL(\mathrm{Spec}^{nc} \circ \widetilde{C}^{*}(\mathfrak{g}))$.

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Checklist of Infinitesimal derived foliation

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- enjoy **formal leaves** described by formal moduli problems✓
- under some finiteness condition, equivalent to a sub-∞-category of **partition Lie algebroids**✓
- Frobenius kernel of $\mathbb{G}_m$ is an interesting example

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