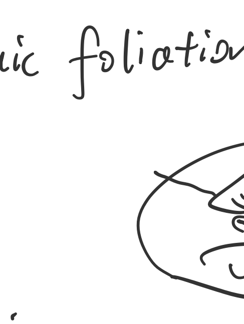
Angers
dimanche 22 septembre 2024 17:42

Formal geometry of foliations

§1 What is an algebraic foliation? T Ω / π $\mathbb{L}$

Background in cplx geom:



$\dim_{\mathbb{C}} = 2$, codim $=1$; $\mathbb{C}^2/\mathbb{Z}^4$ $\alpha\partial_x + \beta\partial_y$

Inspiring example:

- $G \curvearrowright X$ nicely, $\{X/G\}$ space of orbits
- $X \xrightarrow{f} Y$ a holo. fibration

Def A holomorphic foliation $\mathcal{F}$ (codim $= k$) on a cplx. mfd M (dim $= n$) is given by a collection of charts $\{\phi_i : U_i^{\subset M} \to V_i^{\subset \mathbb{C}^k} \times W_i^{\subset \mathbb{C}^{n-k}}\}$

s.t. $\phi_j \circ \phi_i : \mathbb{C}^k \times \mathbb{C}^{n-k} \to \mathbb{C}^k \times \mathbb{C}^{n-k}$, $(x, y) \longmapsto (f(x), g(x,y))$

Thm Frobenius - Nirenberg

There is a 1-1 correspondence
$\{k\text{-codim holo. foliation on } M\} \longleftrightarrow \{T_{\mathcal{F}} \hookrightarrow T_M$ sub bdl, ork $= k$, closed under $[\cdot,\cdot]\}$

Def (T) $X/\mathbb{C}$ sm. var

An algebraic foliation $\mathcal{F}$ is
$$T_{\mathcal{F}} \hookrightarrow T_{X/\mathbb{C}}$$
a saturated qc shf, closed under $[-,-]$.

Q: $X_{reg} :=$ regular loc. of $\mathcal{F}$, $x \in X_{reg}$, $\exists$ a leaf $^{x\in}L \subset X_{reg}$

When L is algebraic. Too difficult, a sta..? in the ..? that ..?

Example $X := \mathrm{Spec}(\mathbb{C}[x, x^{-1}, y])$,
$\mathcal{F}$ is gen. by $\partial_x - \frac{1}{x}\partial_y$
all leaves are like $\{x = ce^{-y}\}$

Def (Ω) $X/\mathbb{C}$ var

An alg. fol $\mathcal{F}$ is a qch quot $\quad$ Ex $\frac{dx}{x} + tdy$
$$\Omega_X \twoheadrightarrow \Omega_{\mathcal{F}}$$
s.t.
$$\begin{array}{ccc} \Omega_X & \xrightarrow{d_{DR}} & \Lambda^2_{\mathcal{O}_X}\Omega_X \\ \downarrow & & \downarrow \\ \Omega_{\mathcal{F}} & \overset{\exists}{\dashrightarrow} & \Lambda^2\Omega_{\mathcal{F}} \end{array}$$

$I := \ker(\Omega_X \twoheadrightarrow \Omega_{\mathcal{F}})$, $I \overset{loc}{=} (\omega_i)_{1\le i\le k}$ then
$$d\omega_i = \sum \alpha_{ij} \wedge \omega_j$$
⇝ ① also works for formal or analytic context ② suitable for calculus

Thm (Malgrange) $\mathcal{O} := \mathbb{C}[[x_1, \ldots, x_n]]$, Ω = diff. germ
$I = (\omega_i)_{1\le i\le k} \subset \Omega$ diff. ideal
$X = V(\omega_1 \wedge \cdots \wedge \omega_k)$
Then I is integrable if
(1) codim $X \ge 3$
or (2) codim $X \ge 2$ & I is formally integrable

[$I \subset \Omega$ is integrable if $\exists\ \mathcal{O}' \to \mathcal{O}$ s.t. $I = \mathcal{O} \cdot d_{DR}(\mathcal{O}')$]

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com. diff. gr. alg (cdga) $k :=$ fld, $\mathbb{Q} \subset k$ — we can paraphrase ..? formulation of alg fol using cdga

a cdga A over k is
- a k-cochain cplx $(A^\bullet, d)$
- a mult: $(a, b) \mapsto ab$ ① unital ② associative
③ $ab = (-1)^{|a||b|} ba$ ④ $d(ab) = da.b + (-1)^{|a|} a.db$

Exam $DR(X/k) := (\Lambda^\bullet \Omega_{X/k}, d_{DR})$

Obsv Given an alg fol $\Omega_X \twoheadrightarrow \Omega_{\mathcal{F}}$
$(DR(X/k) \twoheadrightarrow) \ (\Lambda^\bullet \Omega_{\mathcal{F}}, \bar{d}_{DR})$ is a cdga

Moreover,
$\{\text{alg. fol}/X\}^{(\Omega)} \overset{1:1}{\longleftrightarrow} \{DR(X/k) \twoheadrightarrow (A^\bullet, d)$ cdgas s.t. $A^0 = \mathcal{O}_X$, $A^\bullet = \Lambda^\bullet_{\mathcal{O}_X} A^1\}$

Chevalley - Eilenberg cplx

$\rho : T \to T_{X/\mathbb{C}}$ Lie Algd, the CE cplx $C^*(\rho)$ is
$((\Lambda^\bullet T)^\vee, d_2)$, for $\omega \in (\Lambda^n T)^\vee$, $x_i \in T$ ($0 \le i \le n$)
$$d_2(\omega)(x_0, \ldots, x_n) = \sum_i \pm \rho(x_i)\omega(x_0, \ldots, \hat{x}_i, \ldots, x_n)$$
$$- \sum_{i<j} \pm \omega([x_i, x_j], \ldots)$$

Rk. CE gives 1-1 corr for sm. Lie Algd and sm. alg. fol $\quad T^\vee \wedge T^\vee \to (\Lambda^2 T)^\vee$

Not good for singular

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§2 Derived geom & formal moduli pbm

R = ring $\quad \mathrm{Mod}_R := \mathrm{Ch}_R[\text{quasi-iso}^{-1}]$, ⊗ — Rk. homological degree for int. ho. type
$\mathrm{CAlg}_R := \{E_\infty\text{-algs}/R\}$

when $R/\mathbb{Q}$, $\mathrm{CAlg}_R = \mathrm{cdga}_R[q.\text{iso}^{-1}]$

André - Quillen's $\mathbb{L}$ $\quad R/k$ fin. type,
$R^\bullet \to R$ resol. in poly $\quad \cdots R^2 \substack{\to\\\to\\\to} R^1 \substack{\to\\\to} R^0 \to R$
$$\mathbb{L}_{R/k} := |\Omega_{R^\bullet/k} \otimes_{R^\bullet} R| \in \mathrm{Mod}_R \quad \text{dualizable} \Rightarrow \text{perf}$$
gluing ⇝ $\mathbb{L}_{X/k} \in QC(X)$ for var.

Classical def of var $\quad \mathbb{T}_{X/k} := \mathbb{L}_{X/k}^\vee$
$H^0(X, \mathbb{T}_X) = \{$auto of triv def of X to $k[\varepsilon]\}$
$H^1(X, \mathbb{T}_X) = \{$def of X to $k[\varepsilon]\}$
$H^2(X, \mathbb{T}_X) \ni$ obstruction of def $[k \cdot \varepsilon]$

Def $k/\mathbb{Q}$. k-dg-Lie $\mathfrak{g}$ is
$(\mathfrak{g}^\bullet, d)$ a k-chain cplx
$[-,-] : \mathfrak{g}^\bullet \times \mathfrak{g}^\bullet \to \mathfrak{g}^\bullet$ s.t.
① $[x, y] = (-1)^{|x||y|+1}[y, x]$
② $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|x||y|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$
③ $d[x, y] = [dx, y] + (-1)^{|x|}[x, dy] = 0$

Exam $\mathcal{O}_X \xrightarrow{d_{DR}} \mathbb{L}_X \rightsquigarrow \mathbb{T}_X \to \mathrm{End}(\mathcal{O}_X) \rightsquigarrow \Gamma^\bullet(X, \mathbb{T}_X)$

Def (1) $\mathrm{Art}^*_k \subset \mathrm{cdga}_k$, aug. con alg A s.t.
$\dim_k \pi_*(A) < \infty$
⇝ $\mathcal{X} : \mathrm{Art}^*_k \longrightarrow \mathcal{S}$ fmp s.t.
— $\mathcal{X}(k) \simeq *$
— $\sigma : \begin{array}{ccc} A & \to & A_0 \\ \downarrow & \ulcorner & \downarrow \phi \\ A_1 & \to & A_{01} \end{array}$ s.t. $\pi_0(\phi)$ is surj
⇒ $\mathcal{X}(\sigma)$ is a hmtp pullback

Thm (Lurie - Pridham)
$\mathrm{dg.Lie}_k \simeq \mathcal{FMP}$
$\mathfrak{g} \longmapsto \mathrm{Map}(\mathfrak{D}(-), \mathfrak{g})$

where $\mathfrak{D}(A) = \mathbb{T}_{k/A}$

Def ($\mathbb{T}$) R/k cdga
dg. Lie Algd $\quad \rho : \mathfrak{g} \to \mathbb{T}_{R/k}$ map of R-mod, k-dg-Lie
$[x, ry] = \rho(x)(r)y + (-1)^{|x||r|} r[x, y]$

Thm (Nuiten) R lad
$\mathrm{LieAlgd}^{dg}_{R/k} \simeq \mathcal{FMP}_{\mathrm{Spec}R/}$

§3 Derived fol

Def. $\mathbb{L}$ (Toën - Vezzosi) $\quad X/\mathbb{C}$ Deligne Mumford stack
a graded mixed cdga $(\mathbb{L}\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}}[1]), d)$
$$\mathcal{F} : \mathcal{O}_X \xrightarrow{d} \mathbb{L}_{\mathcal{F}} \xrightarrow{d} \Lambda^2\mathbb{L}_{\mathcal{F}} \to \cdots$$
s.t. $\mathbb{L}_{\mathcal{F}}$ is cnn, perfect

$\mathcal{F}$ is q-cnn if $\mathbb{L}_{\mathcal{F}}$ Tor amplitude $\subset [0, 1)$
rigid if $\Omega_X \twoheadrightarrow \pi_0(\mathbb{L}_{\mathcal{F}}) =: \Omega_{\mathcal{F}}$
if both, $\mathbb{L}_{\mathcal{F}} = \begin{pmatrix} \mathcal{V}^* \\ \downarrow \\ \Omega_X \end{pmatrix}$

Prp (TV) $X/\mathbb{C}$ sm. $\mathcal{F}$ q-cnn + rigid
① $\mathcal{F}$ is locally formally int
② $\mathrm{Sing}(\mathcal{F})$ of codim ≥ 2, $\mathcal{F}$ is locally holo. int

Thm (TV) $X/\mathbb{C}$ cnnced T_2, $\mathcal{F}$ q-cnn + rigid + S then
then $\exists\ X \xrightarrow{\pi} X/\!/\mathcal{F}$ analytic DM 1-truth
s.t. $\mathcal{F} = DR(\pi)$

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§4 Filtered CE cplx

$$C^* : \mathrm{LieAlgd}^{dg}_{R/k} \rightleftarrows (\mathrm{cdga}_{k//R})^{\mathcal{T}} : \mathbb{D}$$
$$C^*(\mathfrak{g}) = (\mathrm{Hom}_R(\mathrm{Sym}\,\mathfrak{g}[1], R), d_1 + d_2)$$
$d_1(\omega) = d \circ \omega - \omega \circ d$
$$d_2(\omega)(x_1, \ldots, x_n) = \sum \pm \rho(x_i)(\omega(\ldots \hat{x}_i \ldots))$$
$$- \sum_{i<j} \pm \omega([x_i, x_j], \ldots)$$

Hodge fil $F^H_n C^*(\mathfrak{g}) = (\mathrm{Hom}_R(\mathrm{Sym}^{\ge n}\mathfrak{g}[1], R), d_1 + d_2)$
$(\mathrm{Gr}^H_\bullet C^*(\mathfrak{g}), d_{DR})$ is a derived fol if
$\mathfrak{g}$ is R-perfect, Tor amplitude $(-\infty, 0]$

Thm (F.) char 0
$$\mathrm{LieAlgd}^{\mathrm{perf}}_{R/k, \le 0} \simeq \{\text{Derived foliations}\}^{\mathcal{T}}$$
S

general B/R cn, $\mathbb{L}_{B/R}$ a perf
$$\mathrm{LieAlgd}^{\mathrm{a\,perf}}_{B/R, \le 0} \simeq (\mathrm{DAlg}^{\mathrm{Fil}}_{B/R, \mathrm{ap}})^{\mathcal{T}}$$

Ekedahl - Shepherd-Barron - Taylor

$X, \mathcal{F}$ to $\mathcal{X}, \mathcal{F}_{\mathcal{X}} \rightsquigarrow \mathcal{X}_{\mathfrak{p}}, \mathcal{F}_{\mathfrak{p}}$
$\mathbb{C} \qquad \mathcal{O}_K[\frac{1}{N}] \qquad k(\mathfrak{p})$ $\quad \partial^{[p]} = \underbrace{\partial \circ \partial \cdots \circ \partial}$ $\quad p = \mathrm{char}\, k(\mathfrak{p})$

——— conjecture:

$[\cdot, \cdot] \circlearrowright \mathfrak{g}_1 \circlearrowleft (-)^{[2]}$
$\downarrow R'$
$[\mathfrak{g}_1, -] \circlearrowright \mathfrak{g}_0$